Measure Theory with Ergodic Horizons
Lecture 17

We will prove all the properties required from an integral shortly, but first we record:

Prop. Let $f, g \in L^+(X,\mu)$.

(a) If $f \le g$ then $\int f \, d\mu \le \int g \, d\mu$.

(b) $\int a \cdot f \, d\mu = a \cdot \int f \, d\mu$ for all $a \in (0,\infty)$.

(c) $\int f \, d\mu = 0$ if and only if $f = 0$ a.e.

Proof. (c). $\Leftarrow$. If $f = 0$ a.e. then every non-negative simple function $s \le f$ also has to be 0 a.e., so $\int s \, d\mu = 0$, hence $\int f \, d\mu = 0$.

$\Rightarrow$. Since $X_0 := \{x \in X : f(x) > 0\} = \bigcup_{n=1}^{\infty} X_n$, where $X_n := \{x \in X : f(x) > \frac{1}{n}\}$, ctbl subadditivity implies that if $\mu(X_0) > 0$, then $\mu(X_n) > 0$ for some $n \ge 1$. But then the simple function $s := \frac{1}{n} \cdot \mathbb{1}_{X_n} \le f$ hence $\int f \, d\mu \ge \int s \, d\mu = \frac{1}{n} \cdot \mu(X_n) > 0$. $\square$

Monotone Convergence Theorem (MCT). Let $f_n, f \in L^+(X,\mu)$. If $f_n \nearrow f$ then $\int f_n \, d\mu \nearrow \int f \, d\mu$.

Proof. Because $f_n \le f$, we have $\int f_n \, d\mu \le \int f \, d\mu$, so we need to show that

$$\lim_n \int f_n \, d\mu \ge \int f \, d\mu,$$

for which it is enough to show that for each non-negative simple function $s \le f$, $\lim_n \int f_n \, d\mu \ge \int s \, d\mu$. Giving ourselves an $\varepsilon > 0$ of room, it's enough to show that $\forall \varepsilon > 0$, $\lim\limits_{n\to\infty} \int f_n \, d\mu \geq \int (1-\varepsilon) s \, d\mu$ for some $n \in \mathbb{N}$.

To this end, note that for each $x \in X$ $\exists\, n_x \in \mathbb{N}$ such that $f_{n_x}(x) > (1-\varepsilon)\cdot s$, thus $X = \bigcup\limits_{n\in\mathbb{N}}^{\uparrow} X_n$, where $X_n := \{x \in X : f_n(x) > (1-\varepsilon)\cdot s\}$. By part (a) above,

$$\int f_n \, d\mu \geq \int_{X_n} f_n \, d\mu \geq \int_{X_n} (1-\varepsilon)\cdot s \, d\mu = \mu_{(1-\varepsilon)s}(X_n)$$

But $\mu_{(1-\varepsilon)s}(X_n) \nearrow \mu_{(1-\varepsilon)s}(X) = \int (1-\varepsilon) s \, d\mu$ by the monotonicity of the measure $\mu_{(1-\varepsilon)s}$, so $\lim\limits_{n\to\infty} \int f_n \, d\mu \geq \int (1-\varepsilon) s \, d\mu$. □

Corollary. The integral on $\mathcal{L}^+(X,\mu)$ is ctbly additive, i.e. $\int \sum\limits_{n\in\mathbb{N}} f_n \, d\mu = \sum\limits_{n\in\mathbb{N}} \int f_n \, d\mu$ for all $f_n \in \mathcal{L}^+(X,\mu)$.

Proof. Firstly, we show that for $f, g \in \mathcal{L}^+(X,\mu)$, we have

$$\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu.$$

Let (f_n) and (g_n) be nonnegative simple functions increasing to f and g, respectively, i.e. $f_n \nearrow f$ and $g_n \nearrow g$. Then $f_n + g_n \nearrow f+g$, so three applications of MCT, we get $\int (f+g) \, d\mu = \lim\limits_n \int (f_n + g_n) \, d\mu = \lim\limits_n \int f_n \, d\mu + \lim\limits_n \int g_n \, d\mu = \int f \, d\mu + \int g \, d\mu$.

Now for an infinite sum $\sum\limits_{n\in\mathbb{N}} f_n$, note that $\sum\limits_{n\leq N} f_n \nearrow \sum\limits_{n\in\mathbb{N}} f_n$ as $N \to \infty$, so the MCT again gives

$$\int \sum_{n\in\mathbb{N}} f_n \, d\mu = \lim_{N\to\infty} \int \sum_{n\leq N} f_n \, d\mu = \lim_{N\to\infty} \sum_{n\leq N} \int f_n \, d\mu = \sum_{n\in\mathbb{N}} \int f_n \, d\mu.$$ □

Corollary. Each $f \in \mathcal{L}^+(X,\mu)$ defines a measure μ_f on Meas_μ by

$$\mu_f(B) := \int_B f\,d\mu := \int f\cdot \mathbb{1}_B\,d\mu$$

for each μ-measurable set $B \subseteq X$.

Proof. Since obviously $\mu_f(\emptyset)=0$, we only need to show ctbl additivity. Let $B = \bigsqcup_{n\in\mathbb{N}} B_n$ be a partition of a μ-measurable set B into μ-measurable sets $B_n \subseteq X$. We need to show $\mu_f(B) = \sum_{n\in\mathbb{N}} \mu_f(B_n)$. But

$$\sum_n \mu_f(B_n) = \sum_n \int f\cdot \mathbb{1}_{B_n}\,d\mu = \int \sum_n f\cdot \mathbb{1}_{B_n}\,d\mu = \int f \sum_n \mathbb{1}_{B_n}\,d\mu = \int f\cdot \mathbb{1}_B\,d\mu = \mu_f(B). \quad \square$$

Corollary (Fatou's lemma). Let $(f_n) \subseteq L^+(X,\mu)$. Then

$$\int \liminf_{n\to\infty} f_n\,d\mu \leq \liminf_{n\to\infty} \int f_n\,d\mu.$$

Proof. Recall that for reals $x_n \in [-\infty,\infty]$, $\liminf\limits_{n\to\infty} x_n := \lim\limits_{n\to\infty} \inf\{x_m : m\geq n\}$.

Note that the sequence $\inf\{x_m : m \geq n\}$ is increasing. By MCT,

$$\int \liminf_{n\to\infty} f_n\,d\mu = \int \lim_n \inf\{f_m : m\geq n\}\,d\mu = \lim_n \int \inf\{f_m : m\geq n\}\,d\mu.$$

Lastly, note that for each $n\in\mathbb{N}$, $\inf\{f_m : m\geq n\} \leq f_m$ for all $m \geq n$.

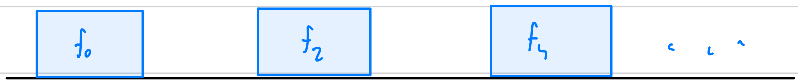
So by monotonicity, $\int \inf\{f_m : m\geq n\}\,d\mu \leq \int f_m\,d\mu$ for all $m\geq n$, hence

$$\int \inf\{f_m : m\geq n\}\,d\mu \leq \inf\left\{\int f_m\,d\mu : m\geq n\right\}.$$

Thus, $\lim\limits_{n\to\infty} \int \inf\{f_m : m\geq n\}\,d\mu \leq \lim\limits_{n\to\infty} \inf\left\{\int f_m\,d\mu : m \geq n\right\} = \liminf\limits_{n\to\infty} \int f_n\,d\mu$. $\square$

Examples (of strict inequality in Fatou's lemma). Let $(X, \mu) := (\mathbb{R}, \lambda)$.

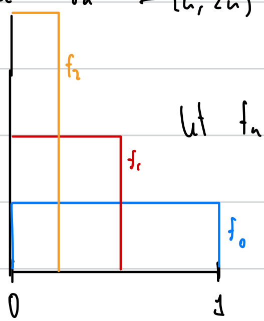
(a) Let $f_n := \mathbb{1}_{[n, n+1)}$, then $f_n \to 0$ pointwise, but $\int f_n \, d\lambda = 1$ for all $n \in \mathbb{N}$.



(a') Let $f_n := \mathbb{1}_{[n, \infty)}$, then $f_n \to 0$ pointwise, but $\int f_n \, d\lambda = \infty$ for all $n \in \mathbb{N}$.

(a'') Let $f_n := \mathbb{1}_{[n, 2n)}$, then $f_n \to 0$ pointwise, but $\int f_n \, d\lambda = n$ for all $n \in \mathbb{N}$.

(b)



Let $f_n := n \, \mathbb{1}_{(0, \frac{1}{n}]}$, $n \geq 1$. Then again $f_n \to 0$ but $\int f_n \, d\lambda = 1$ $\forall n$.

(b') Let $f_n := n^2 \, \mathbb{1}_{(0, \frac{1}{n}]}$, $n \geq 1$. Then $f_n \to 0$ but $\int f_n \, d\lambda = n^2 \cdot \frac{1}{n} = n \to \infty$.

Def. A μ-measurable function $f : X \to \mathbb{R}$ is called *μ-integrable* if $\int |f| \, d\mu < \infty$.
The *integral* of such f is defined to be

$$\int f \, d\mu := \int f_+ \, d\mu - \int f_- \, d\mu.$$

The set of all integrable functions is denoted by $L^1(X, \mu)$.

Note that $L^1(X, \mu)$ is a vector space and we make it into a pseudo-normed vector space by equipping it with the following pseudo-norm:

$$\|f\|_1 := \int |f| \, d\mu.$$

<u>Observation.</u> $\|\cdot\|_1$ is indeed a pseudo-norm on $L^1(X,\mu)$, i.e. for all $f,g \in L^1(X,\mu)$:

(i) $\|f\|_1 \geq 0$ and $\|f\|_1 = 0 \iff f = 0$ a.e. (this a.e. is why it's not a norm).

(ii) $\|\alpha f\|_1 = |\alpha| \cdot \|f\|_1$ for all $\alpha \in \mathbb{R}$.

(iii) Triangle inequality: $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$.

Proof. For (iii), note that $\|f+g\|_1 = \int |f+g| \, d\mu \leq \int (|f|+|g|) \, d\mu = \int |f| \, d\mu + \int |g| \, d\mu = \|f\|_1 + \|g\|_1$. □

This pseudo-norm also defines a metric on $L^1(X,\mu)$ by

$$d_1(f,g) := \|f-g\|_1,$$

making $L^1(X,\mu)$ into a pseudo-metric space. Thus, it makes sense to say that a sequence $(f_n) \subseteq L^1(X,\mu)$ converges in the pseudo-norm $\|\cdot\|_1$ to f to mean that $\|f_n - f\|_1 \to 0$. We denote this by $f_n \to_{L^1} f$.